Hopcroft–Karp Algorithm

An $n^{5/2}$ Algorithm for Maximum Matchings in Bipartite Graphs

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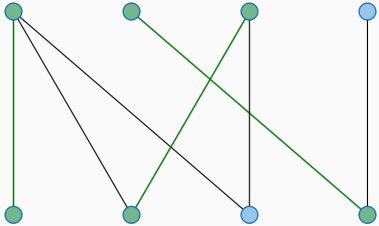

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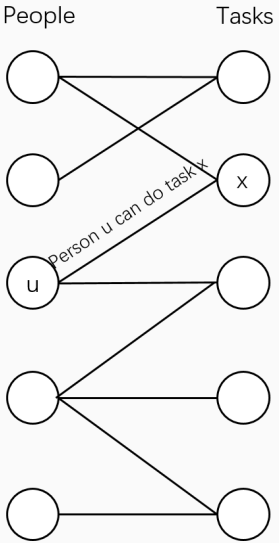
Introduction

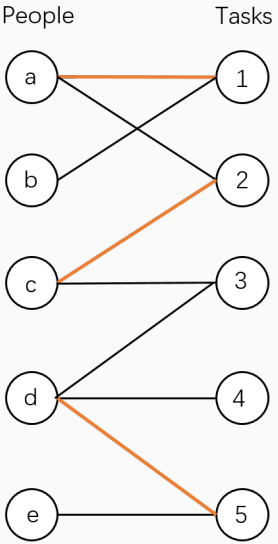
Bipartite Graphs

- Suppose we have a set of people L and set of tasks R
- Each person can do only some of the tasks
- Can model this as a bipartite graph $\rightarrow$



Bipartite Matching

- A **matching** gives an assignment of people to tasks
- Want to get as many tasks done as possible
- So, want a **maximum matching**: one that contains as many edges as possible.
- (This one is not maximum.)



Hopcroft–Karp Algorithm

Figure 1: John Hopcroft. 1986, Turing Award

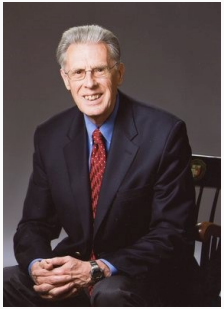


Figure 2: Richard M. Karp. 1985, Turing Award

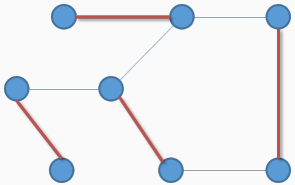
- Hungarian algorithm: time complexity $O(|E||V|)$
- Hopcroft–Karp algorithm: time complexity $O(|E|\sqrt{|V|})$
- Alt et al. algotithm:$O(|V|^{1.5}\sqrt{|E|/log|V|})$

Matchings and Augmenting Paths

Matching

Definition

- Let $G = (V, E)$ be a finite *undirected* graph (without loops, multiple edges, or isolated vertices)
- A set $M \subseteq E$ is a matching if **no** vertex $v \in V$ is incident with more than one edge in M
- Maximum matching is a matching of maximum cardinality
- A vertex v is free if it is incident with **no** edge in M



Augmenting Path

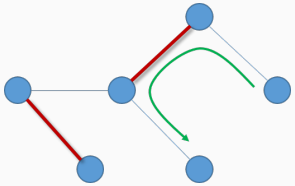
Definition

- A path (without repeated vertices)

$$P = (v_1, v_2), (v_2, v_3), \cdots, (v_{2k-1}, v_{2k})$$

  is called an augmenting path if its endpoints v_1 and v_{2k} are both free, and its edges are alternatively in $E - M$ and in M

- P has an odd number of edges
- $|P \cap (E - M)| = |P \cap M| + 1$
- $|P| = |P \cap (E - M)| + |P \cap M| = 2|P \cap M| + 1$



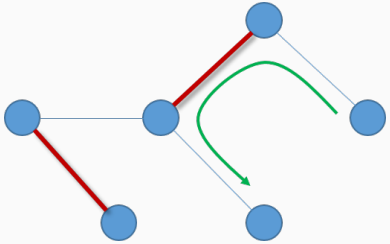
Lemma 1

Lemma

If M is a matching and P is an augmenting path relative to M, then $M \oplus P$ is a matching, and $|M \oplus P| = |M| + 1$.

Proof.

- $|P| = 2|P \cap M| + 1$
- $|M \oplus P| = |M| + |P| - 2|P \cap M|$
- $|M \oplus P| = |M| + 1$
- vertices in the complement of P have the same neighbors in M as in $M \oplus P$, and vertices in P have **exactly** one neighbor in $M \oplus P$



□

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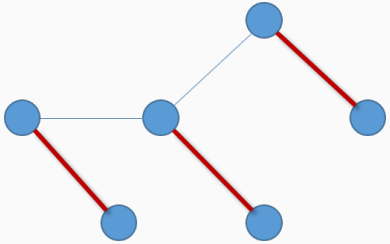

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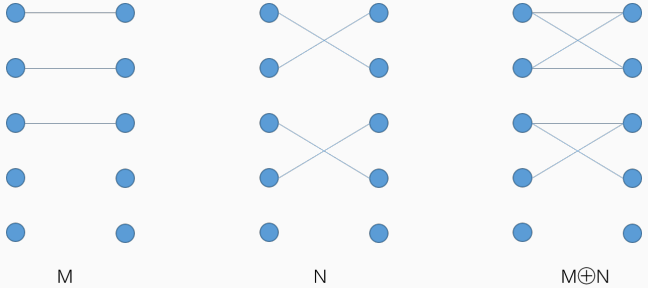
Theorem 1

Theorem

Let M and N be matchings. If $|M| = r$, $|N| = s$ and $s > r$, then $M \oplus N$ contains ***at least*** *$s - r$ vertex-disjoint augmenting paths relative to M.*



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Let M and N be matchings. If $|M| = r$, $|N| = s$ and $s > r$, then $M \oplus N$ contains ***at least*** *$s - r$ vertex-disjoint augmenting paths relative to M.*

Proof.

- $\overline{G} = (V, M \oplus N)$
- $\forall v \in V$, v indicent with **at most one** edge from $N - M$ and $M - N$ respectively
- each connected component of $\overline{G}$ is either
  1. an isolated vertex
  2. a cycle of **even** length, with edges alternatively in $M - N$ and in $N - M$
  3. a path whose edges are alternatively in $M - N$ and in $N - M$

□

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Let M and N be matchings. If $|M| = r$, $|N| = s$ and $s > r$, then $M \oplus N$ contains ***at least*** *$s - r$ vertex-disjoint augmenting paths relative to M.*

Proof.

- Let the components of $\overline{G}$ be $C_1, C_2, \cdots, C_g$, where $C_i = \{V_i, E_i\}$
- let $\delta(C_i) = |E_i \cap N| - |E_i \cap M| \in \{-1, 0, 1\}$
- $\delta(C_i) = 1$ **iff** C_i is an augmenting path relative to M

$$\begin{aligned}
\sum_i \delta(C_i) &= \sum_i (|E_i \cap N| - |E_i \cap M|) = \sum_i |E_i \cap N| - \sum_i |E_i \cap M| \\
&= |\bigcup_i E_i \cap N| - |\bigcup_i E_i \cap M| = |M \oplus N \cap N| - |M \oplus N \cap M| \\
&= |N - M| - |M - N| = (|N - M| + |M \cap N|) - (|M - N| + |M \cap N|) \\
&= |N| - |M| = s - r
\end{aligned}$$

Corollary 1

Corollary

*M is a maximum matching **iff** there is no augmenting path relative to M.*

Proof.

maximum matching $\Rightarrow$ no augmenting path

- Lemma 1

no augmenting path $\Rightarrow$ maximum matching

- Suppose that M^* is a matching of maximum cardinality, and that $|M| < |M^*|$
- $M \oplus M^*$ contains at least 1 vertex-disjoint augmenting paths relative to M

□

Naive Algorithm

Algorithm 1: Naive iterative scheme for computing a maximum matching

$M = \emptyset$

repeat

  Find an augmenting path P with respect to M

  $M = M \oplus P$

until *there is no augmenting with respect to M*

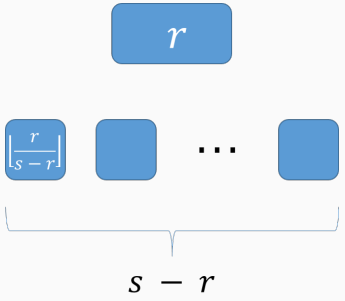
Corollary 2

Corollary

Let M be a matching. Suppose $|M| = r$, and suppose that the cardinality of a ***maximum*** *matching is s, $s > r$. Then there exists an augmenting path relative to M of length $\leq 2\lfloor r/(s-r) \rfloor + 1$.*

Proof.

- Let N be a **maximum** matching
- $M \oplus N$ contains $s - r$ vertex-disjoint augmenting paths relative to M
- **at most** r edges from $M \Rightarrow$ *one of them* **at most** $\lfloor r/(s-r) \rfloor$ edges from M
- **at most** $\leq 2\lfloor r/(s-r) \rfloor + 1$ *all altogether*



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Theorem 2

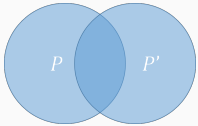
Theorem

Let M be a matching, P a shortest augmenting path relative to M, and P' an augmenting path relative to $M \oplus P$. Then

$$|P'| \geq |P| + |P \cap P'|$$

Proof.

- Let $N = M \oplus P \oplus P'$. Then N is a matching and $|N| = |M| + 2$
- $M \oplus N$ contains 2 vertex-disjoint augmenting paths relative to M; call them P_1 and P_2
- $M \oplus N = M \oplus (M \oplus P \oplus P') = P \oplus P' \Rightarrow |P \oplus P'| \geq |P_1| + |P_2| \geq 2|P|$
- $|P \oplus P'| = |P| + |P'| - 2\,|P \cap P'| \Rightarrow |P'| \geq |P| + 2|P \cap P'|$ (Warning!)



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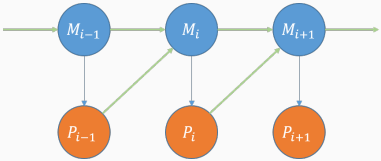
Notations

- Initially $M_0 = \emptyset$
- Compute a sequence $M_0, M_1, M_2, \cdots, M_i, \cdots$
- P_i is a shortest augmenting path relative to M_i
- $M_{i+1} = M_i \oplus P_i$



Corollary

$|P_i| \leq |P_{i+1}|$

Proof.

P_{i+1} is a augmenting path relative to $M_i \oplus P_i$, then

$$|P_{i+1}| \geq |P_i| + |P_i \cap P_{i+1}| \geq |P_i|$$

□

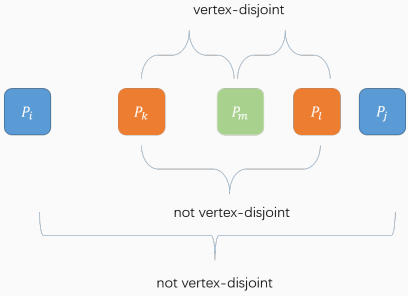
Corollary 4

Corollary

For all i and j such that $|P_i| = |P_j|, (i \neq j)$, P_i and P_j are vertex-disjoint.

Proof.

- By contradiction, $|P_i| = |P_j|, i < j, P_i$ and P_j are **not** vertex-disjoint.
- $\exists k, l$ such that $i \leq k < l \leq j$, P_k and P_l are **not** vertex-disjoint **and** $\forall m \in (k, l)$, P_m is vertex-disjoint from P_k and P_l



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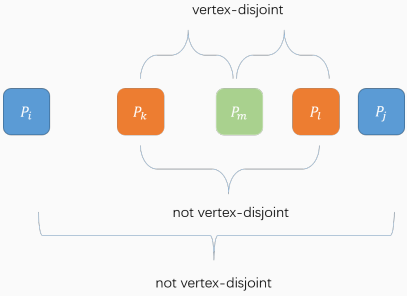
Corollary 4

Corollary

For all i and j such that $|P_i| = |P_j|$, $(i \neq j)$, P_i and P_j are vertex-disjoint.

Proof.

- P_l is an augmenting path relative to $M_k \oplus P_k$
  - $\forall m \in (k, l)$, P_m is vertex-disjoint from P_k and P_l
- $\Rightarrow |P_l| \geq |P_k| + |P_k \cap P_l|$
- But $|P_l| = |P_k|$, so $|P_k \cap P_l| = 0$
  $\Rightarrow P_k$ and P_l have no edges in common
- if P_k and P_l had a vertex v in common
  - they would have a common edge in $M_k \oplus P_k$



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Theorem 3

Theorem

Let s be the cardinality of a maximum matching. The number of distinct integers in the sequence

$$|P_0|, |P_1|, \cdots, |P_i|, \cdots$$

is less than or equal to $2\lfloor\sqrt{s}\rfloor + 2$.

Proof.

- Let $r = \lfloor s - \sqrt{s}\rfloor$. Then $|M_r| = r$ and, by Corollary 2,

$$|P_r| \leq 2\lfloor s - \sqrt{s}\rfloor/(s - \lfloor s - \sqrt{s}\rfloor) + 1 \leq 2\lfloor\sqrt{s}\rfloor + 1$$

- for each $i < r$, $|P_i|$ is one of the $\lfloor\sqrt{s}\rfloor + 1$ positive odd integers $\leq 2\lfloor\sqrt{s}\rfloor + 1$
- Also $|P_{r+1}|, \cdots, |P_s|$ contribute at most $\lceil\sqrt{s}\rceil$ distinct integers
- total $\leq \lfloor\sqrt{s}\rfloor + 1 + \lceil\sqrt{s}\rceil \leq 2\lfloor\sqrt{s}\rfloor + 2$

□

Maximum Matching Algorithm

Algorithm 2: Maximum matching algorithm

$M = \emptyset$

repeat

$l =$ the length of a **shortest** augmenting path relative to M

Find a **maximal** set of paths $\{Q_1, Q_2, \cdots, Q_t\}$ with the properties that

1. $\forall Q_i, Q_i$is an augmenting path relative to M and $|Q_i| = l$
2. the Q_i are vertex-disjoint

$M = M \oplus Q_1 \oplus Q_2 \oplus \cdots \oplus Q_t$

until *no such path exists*

Corollary 5

Corollary

If the cardinality of a maximum matching is s, then Algorithm constructs a maximum matching within $2\lfloor\sqrt{s}\rfloor + 2$ *executions of Find step.*

Proof.

Consider the ith and (i+1)th executions of Find steps.

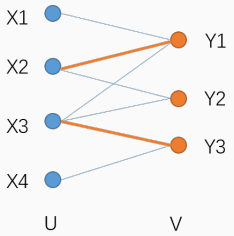
- $l_{i+1} \geq l_i$, by Corollary 3
- $l_{i+1} = l_i \Rightarrow \forall P_a \in phase_i, P_b \in phase_{i+1}$ P_a and P_b are vertex-disjoint

□

The Bipartite Case

Algorithm

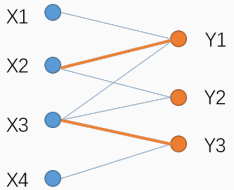
- let *U* and *V* be the two sets in the bipartition of *G*
- let the matching from *U* to *V* at any time be represented as the set *M*
- The algorithm is run in phases. Each phase consists of the following steps.
  - A breadth-first search partitions the vertices of the graph into layers.
  - finds a maximal set of vertex disjoint augmenting paths of minimum length by *DFS*
  - Use these augmenting paths to enlarge *M*

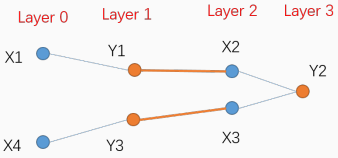


Partitions the Vertices

A breadth-first search partitions the vertices of the graph into layers.

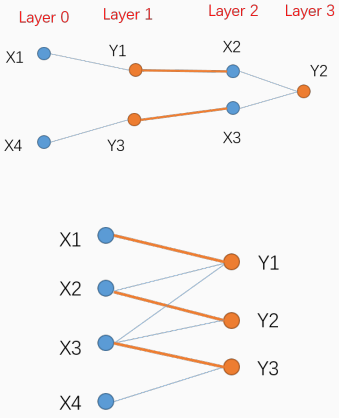
- The free vertices in U form the first layer of the partitioning.
- the traversed edges are required to alternate between matched and unmatched.
  - from a vertex in U, only unmatched edges may be traversed.
  - from a vertex in V only matched edges may be traversed.
- The search terminates at the first layer k where one or more free vertices in V are reached.





Maximal Set of Vertex Disjoint Augmenting Paths

- finds a *maximal* set of vertex disjoint augmenting paths of length k
  - the DFS is only allowed to follow edges that lead to an unused vertex in the next layer
  - paths in the DFS must alternate between matched and unmatched edges
  - Any vertex encountered during the DFS can immediately be marked as used.
- $O(|E|)$ running time for the DFS
- Every one of the paths found in this way is used to enlarge M



Hopcroft-Karp Algorithm

Algorithm 3: Hopcroft-Karp Algorithm

```
1 while exist augmenting paths
2     BFS(G, M)
3     used[] = false
4     for each free boy vertex u
5         DFS(G, M, u)
```

Construct Graph

Algorithm 4: BFS(G,M)

```
1  Create empty queue Q
2  Push all free boy vertices into Q
3  while Q ≠ ∅
4      u = DEQUEUE(Q)
5      if u.type= boy
6          for each v ∈ G.Adj[u]
7              if not visited v
8                  v.dist = u.dist + 1
9                  If not free_girl_found ENQUEUE(Q, v)
10     else if u.type= girl
11         if ∃(u, v) ∈ M
12             v.dist = u.dist + 1
13             ENQUEUE(Q, v)
14         else free_girl_found = true
```

Finding a matching for u

Algorithm 5: Depth-First-Search

```
1 Function DFS( G, M, u )
2     for each v ∈ G.Adj[u]
3         if not used[v] and v.dist = u.dist + 1
4             used[v] = true
5             if v not matched or DFS(G, M, the current macthing of v)
6                 M = M ∪ {(u, v)}
7                 return true
8     return false
```

Conclusion

- Time Complexity
  - a single phase may be implemented in $O(|E|)$ time.
  - the algorithm performs a total of at most $2\sqrt{|V|}$ phases
  - it takes a total time of $O(|E|\sqrt{|V|})$ in the worst case.
- Non-bipartite graphs
  - the task of finding the augmenting paths within each phase is more difficult
  - Micali, S.; Vazirani, V. V. (1980); "An $O(\sqrt{|V||E|})$ algorithm for finding maximum matching in general graphs"
  - Peterson, Paul A.; Loui, Michael C. (1988), "The general maximum matching algorithm of Micali and Vazirani"

Questions?

Thanks.